

DIVISIBILITY THEORY AND COMPLEXITY OF ALGORITHMS FOR FREE PARTIALLY COMMUTATIVE GROUPS

EVGENII S. ESYP, ILYA V. KAZACHKOV, AND VLADIMIR N.
REMESLENNIKOV

NOTA BENE. The original version of the paper was published in Contemporary Mathematics 378 “Groups, Languages, Algorithms”; 2005, pp. 319-348. This is a modified version with Appendix that holds a corrected formulation of Proposition 4.1.

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The second author was partially supported by the London Mathematical Society and by EPSRC grant GR/S71200.

The third author was supported by EPSRC grants GR/R29451 and GR/S71200.

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1. INTRODUCTION

Free partially commutative groups (or *partially commutative groups*, for brevity) have many remarkable properties.

- They arise naturally in many branches of mathematics and computer science. This led to a variety of names under which they are known: *semifree groups* [2, 3], *graph groups* [26, 55, 56, 57], *right-angled Artin groups* [10, 19], *locally free groups* [20, 58].
- Normal forms of elements in these groups are extremely convenient from computational point of view.
- All major algorithmic problems for partially commutative groups are solvable.
- The groups have very rich subgroup structure. In particular, the fundamental group of almost every surface is a subgroup of a suitable partially commutative group (see [19]).
- Free partially commutative groups have interesting statistical properties, and many of their statistical parameters can be estimated.
- Right-angled groups are intrinsically connected to other types of Artin groups [15, 53].

Not surprisingly, the rich pickings of the theory of partially commutative groups attract researchers from various areas of mathematics, and

the group are being investigated from a refreshing variety of points of view.

Without attempting to give a full survey of the existing theory, we mention that the structural theory of partially commutative groups was developed in [36, 56] and structure of subgroups studied in [19, 55]. Paper [10] discusses applications to geometry and topology, while major statistical characteristics for these groups are estimated in [20, 58, 59] and the Poisson-Furstenberg boundary described in [50]. Some algorithmic problems for right angled groups were solved in [38, 62].

We can summarize the main results of this paper as follows. Let G be an arbitrary partially commutative group.

- We develop divisibility theory for G (and not just for semigroups of positive elements, as in the classical paper on Artin groups [11]). It closely resembles the classical divisibility theory of integers. In particular, we have reasonable concepts of the least common multiple and greatest common divisor.
- We prove the existence of polynomial algorithms for solving main algorithmic problems and estimate the boundaries for their complexity.
- With the help of the algorithms of divisibility theory in each class w^G of all elements conjugate to the element w we compute the canonical representative. Its normal form is called in the paper the *minimal exhausted form* of w^G . This form is easy from the computational point of view (see Outline of this algorithm in Section 5.3).
- These results allow us to present a solution of time complexity $O(n^3)$ for the conjugacy problem for partially commutative groups (see Theorem 6.11) and the algorithm in Section 5.4.

In Sections 3 and 5 we describe the algorithms for solving all major problems in partially commutative groups: divisibility problem, computation of the greatest common divisor and the least common multiple of a pair of elements, computation of maximal divisor related to fixed subgroup, computation of block decomposition for an element of partially commutative group, algorithms for computation of normal forms and solving the conjugacy problem.

The principle motivation for the present paper was the desire to understand why genetic [18] and semi-heuristic [40] algorithms happened to be so efficient for solving conjugacy search problem and some other algorithmic problems on special classes of partially commutative groups. See [7] for a brief discussion of genetic algorithms. The analysis shows that the factors behind the good performance of these

semi-deterministic algorithms are manifestation of a good divisibility theory. It also helps us to show, in the final section of the paper, that our deterministic algorithms have at most cubic time complexity.

2. FREE PARTIALLY COMMUTATIVE GROUPS

2.1. Definitions. From now on $X = \{x_1, \dots, x_r\}$ always stands for a finite alphabet, its elements being called *symbols*. In what follows, $x_i, x_j, \dots, y_i, y_j, \dots$, etc. are reserved for symbols from X or their inverses.

Recall that a (free) *partially commutative* group is a group given by generators and relations of the form

$$(1) \quad \mathbb{G} = \langle X \mid R_X \rangle \text{ for } R_X \text{ a subset of } \{[x_i, x_j] \mid x_i, x_j \in X\}$$

This means that the only relations imposed on the generators are commutation of some of the generators. The number $r = |X|$ is called the *rank* of \mathbb{G} . In particular, the free abelian group on X is a partially commutative group. We also denote

$$R = \{[x_i, x_j] \mid x_i, x_j \in X\},$$

and, for $Y \subset X$,

$$R_Y = \{[x_i, x_j] \mid x_i, x_j \in Y\} \cap R_X.$$

Set $X_{r-1} = \{x_2, \dots, x_r\}$. It is easy to see that every partially commutative group $\mathbb{G}_r = \langle X_r \mid R_r \rangle$ is an *HNN*-extension of $\mathbb{G}_{r-1} = \langle X_{r-1} \mid R_{r-1} \rangle$ by the element x_1 . In this extension, associated subgroups H and K coincide and are subgroups of \mathbb{G}_{r-1} generated by all symbols $x_j \in X$ such that x_1 commutes with x_j and $j \neq 1$. The corresponding isomorphism $\phi : H \rightarrow K$ is the identity map $\text{id} : H \rightarrow H$. Indeed, this can be seen from the obvious presentation for \mathbb{G} :

$$\mathbb{G} = \langle \mathbb{G}_{r-1}, x_1 \mid \text{rel}(\mathbb{G}_{r-1}), [x_1, x_j] = 1 \text{ for } [x_1, x_j] \in R_X \rangle$$

Two extreme cases of this construction are worth mentioning. If $H = K$ is the trivial subgroup then G is the free product

$$\mathbb{G} = \mathbb{G}_{r-1} * \langle x_1 \rangle$$

of two partially commutative groups. From the algorithmic point of view, free products of groups behave nicely and most decision problems can be reduced to similar problems for the factors.

Another extreme case, $H = K = \mathbb{G}_{r-1}$, yields the direct product

$$\mathbb{G} = \mathbb{G}_{r-1} \times \langle x_1 \rangle.$$

Again, all algorithmic problems considered in this paper can be easily reduced to the corresponding problems for the direct factors.

$X = \{x_1, \dots, x_r\}$	— a finite alphabet
\mathbb{G}_r , or $\mathbb{G}(X)$, or \mathbb{G}	— free partially commutative group of rank r generated by the set X
\mathbb{G}_r^+ , or $\mathbb{G}^+(X)$, or \mathbb{G}^+	— free partially commutative monoid of rank r generated by the set X
$ w $	— the length of the word w
$[w]$	— the element of the group \mathbb{G}_r represented by the word w
$u = v$	— equality of the elements in the group \mathbb{G}_r
$u \simeq v$	— equality of words in the free monoid $M(X \cup X^{-1})$
$\lg(w)$	— the length of a geodesic word w' such that $w =_{\mathbb{G}_r} w'$
$w = g \circ h$	— cancellation-free multiplication, that is, $\lg(w) = \lg(g) + \lg(h)$.
$n(w)$	— the normal form of w
$RC(w)$	— cyclically reduced conjugate to w
$\alpha(w)$	— the set of symbols which occur in the word w
$\mathbb{A}(w)$	— the subgroup of \mathbb{G} generated by all symbols which commute with w and do not belong to $\alpha(w)$
$D_l(w)$	— the set of left divisors of w
$u \mid v$	— left divisibility, u divides v
$\text{ad}(w)$	— the the maximal (left) abelian divisor of w
$w[i, j]$	— the interval of the word w from the i -th letter and up to and including the j -th one, $i \leq j$
$w[i]$	— the i -th letter in the word w
$e_Y(w)$	— the exhausted form of w with respect to the set $Y \subseteq X$
$\text{gd}(u, v)$	— the greatest common left (right) divisor of the geodesic words u and v
$\text{lm}(u, v)$	— the left (right) least common multiple of the geodesic words u and v
$p_v(w)$	— the greatest left (right) divisor of w such that $\alpha(v)$ and $\alpha(p_v(w))$ commute elementwise, $[\alpha(v), \alpha(p_v(w))] = 1$
$\text{gd}_Y(w)$	— the greatest left (right) divisor of the geodesic word w which belongs to the subgroup $\mathbb{G}(Y)$

FIGURE 1. The table of notation

Proposition 2.1.

- (1) *Free and direct products of partially commutative groups are also partially commutative groups.*
- (2) *Let $Y \subset X$ and $\mathbb{G}(Y) = \langle Y \rangle$. Then $\mathbb{G}(Y) = \langle Y \mid R_Y \rangle$. Therefore the group $\mathbb{G}(Y)$ is also a partially commutative group.*

The subgroups $\mathbb{G}(Y)$ for $Y \subset X$ will be called *parabolic*.

Proof. (1) follows directly from the definitions of free and direct products. For the proof of (2) assume that $X = Y \sqcup \{z_1, \dots, z_k\}$. Notice that $\mathbb{G}(X)$ is an *HNN*-extension of the group $\mathbb{G}(X \setminus \{z_1\})$, while $\mathbb{G}(X \setminus \{z_1\})$ is an *HNN*-extension of the group $\mathbb{G}(X \setminus \{z_1, z_2\})$, etc. Now (2) follows by induction. \square

Notice that $\mathbb{G}(Y)$ is a *retract* of \mathbb{G} , that is, the image of \mathbb{G} under the idempotent homomorphism

$$\begin{aligned} \mathbb{G} &\rightarrow \mathbb{G}(Y) \\ y &\mapsto y && \text{if } y \in Y \\ x &\mapsto 1 && \text{if } x \in X \setminus Y \end{aligned}$$

2.2. ShortLex and normal forms.

2.2.1. Words and group elements. We work mostly with words in alphabet $X \cup X^{-1}$ and distinguish between the equality of words which denote by symbol \simeq , and the equality of elements in the group \mathbb{G} , which we denote $=$. However, we use the convention: when w is a word and we refer to it as if it is an element of \mathbb{G} , we mean, of course, the element $[w]$ represented by the word w . We take special care to ensure that it is always clear from the context whether we deal with words or elements of the group.

2.2.2. SHORTLEX. Our first definition of normal forms involves the SHORTLEX ordering. We order the symmetrised alphabet $X \cup X^{-1}$ by setting

$$x_1 \leq x_1^{-1} \leq x_2 \leq \dots \leq x_r \leq x_r^{-1}.$$

This ordering gives rise to the SHORTLEX ordering \leq of the set of all words (free monoid) in the alphabet $X \cup X^{-1}$: we set $u \leq v$ if and only if either

- $|u| < |v|$, or
- $|u| = |v|$ and u precedes v in the sense of the right lexicographical order.

SHORTLEX is a total ordering of $M(X \cup X^{-1})$. If w is a word and $[w]$ is the class of all words equivalent to w in \mathbb{G}_r , we define w^* as the \leq -minimal representative of $[w]$. The element w^* is called the (SHORTLEX) *normal* form of the word w .

2.2.3. *The Bokut-Shiao rewriting rules.* Bokut and Shiao [5] found a complete set of Knuth-Bendix rewriting rules for the group \mathbb{G}_r :

$$(2) \quad x_i^\epsilon x_i^{-\epsilon} \rightarrow 1$$

$$(3) \quad x_i^\epsilon w(x_{k_1}, \dots, x_{k_t}) x_j^\eta \rightarrow x_j^\eta x_i^\epsilon w(x_{k_1}, \dots, x_{k_t})$$

Here, as usual, $\epsilon, \eta = \pm 1$. The first rule is applied for all $i = 1, \dots, r$, the second one for all symbols x_i, x_j and x_{k_1}, \dots, x_{k_t} and all words $w(x_{k_1}, \dots, x_{k_t})$ such that $i > j > k_s$ while $[x_i, x_j] = 1$, $[x_j, x_{k_s}] = 1$ and $[x_i, x_{k_s}] \neq 1$.

In the case of free partially commutative monoids the rules (3) suffice, cf. Anisimov and Knuth [1].

Note that these transformation are decreasing with respect to the SHORTLEX order. Bokut and Shiao showed every word w can be transformed into its SHORTLEX normal form w^* using a sequence of transformations (2) and (3),

$$(4) \quad w = w_1 \rightarrow \dots \rightarrow w_k = w^*.$$

2.2.4. *Geodesic words and the Cancellation Property.* A *geodesic* form of a given element $w \in \mathbb{G}_r$ is a word that has minimal length among all words representing w . A word is *geodesic* if it is a geodesic form of the element it represents.

Lemma 2.2 (Cancellation Property). *Let w be a non-geodesic word. Then there exists a subword $yw[i, j]y^{-1}$ where $y \in X \cup X^{-1}$ and y commutes with every symbol in the interval $w[i, j]$*

Proof. The lemma follows from rewriting rules (2) and (3) by induction on the length k of a sequence of transformations (4). \square

Lemma 2.3 (Transformation Lemma). *Let w_1 and w_2 be geodesic words which represent the same element of \mathbb{G}_r . Then we can transform the word w_1 into w_2 using only the commutativity relations from R_X .*

Such transformations of geodesic words will be called *admissible transformations* or *admissible permutations*.

Proof. First of all, observe that the SHORTLEX normal form is geodesic. Indeed, the transformation of word into a normal form does not increase

the length of the word. Now, either w_1 or w_2 can be transformed to normal form using rewriting rules (2) and (3). But w_1 and w_2 are geodesic, therefore the lengths of w_1 and w_2 can not decrease. Since the rule (2) decreases the length, this means that we use only the commutativity rules (3), which, in their turn, can be obtained by as a reversible sequence of application of commutativity relations from R_X . \square

2.2.5. Consequences of the Transformation Lemma. Our first observation is that the monoid \mathbb{G}^+ generated in \mathbb{G} by the set X (without taking inverses!) is the free partially commutative monoid on the set X in the sense that it is given by the same generators and relations as \mathbb{G} , but in the category of semigroups.

With the help of Transformation Lemma 2.3, many results in this paper will be proven, and definitions given, first for geodesic words, and then transferred to elements in \mathbb{G}_r because they do not depend on a particular choice of a geodesic word representing the element. For example, if w is a geodesic word, we define $\alpha(w)$ as the set of symbols occurring in w . It immediately follows from Lemma 2.3 that if v is another geodesic word and $u = w$ in \mathbb{G}_r then $\alpha(w) = \alpha(v)$. This allows us to define sets $\alpha(w)$ for elements in \mathbb{G}_r . Notice that

Lemma 2.4. *A symbol $x \in X$ commutes with $w \in \mathbb{G}_r$ if and only if x commutes with every symbol $y \in \alpha(w)$.*

We set $\mathbb{A}(w)$ to be the subgroup of \mathbb{G}_r generated by all symbols from $X \setminus \alpha(w)$ that commute with w (or, which is the same, with $\alpha(w)$).

Notice that we call elements of our alphabet X *symbols*. We reserve the term *letter* to denote an occurrence of a symbol in a geodesic word. In a more formal way, a letter is a pair (symbol, its placeholder in the word). For example, in the word $x_1x_2x_1$ the two occurrences of x_1 are distinct letters. If $Y \subset X$ and x a letter, we shall abuse notation and write $x \in Y$ if the symbol of the letter x belongs to Y . This convention allows us to use the full strength of Transformation Lemma 2.3 and talk about application of rewriting rules (4) to geodesic words as a rearrangement of the letters of the word. It is especially helpful in the discussion of divisibility (Section 3).

The following two results are obvious corollaries of the Transformation Lemma.

Lemma 2.5. *A word u can be transformed to any of its geodesic forms using only cancellations (2) and permutations of letters allowed by the commutativity relations from R_X , that is, rules (3) and their inverses.*

We shall call such transformations of words *admissible*.

Lemma 2.6. *If x and y are two letters in a geodesic word w , and $[x, y] \neq 1$, then admissible permutations of letters in w do not change the relative position of x and y : if x precedes y , then it does so after a permutation.*

Lemma 2.7. *If u and v are geodesic words. The only cancellations which take place in the process of an admissible transformation of the word uv into a geodesic form are those which involve a letter from u and a letter from v .*

Proof. Colour letters from u red and letters from v black. Assume the contrary, and suppose that two black letters x and y cancel each other in the process of admissible transformation of uv into a geodesic form. We can choose x and y so that they are the first pair of black letters cancelled in the process. Assume also that x precedes y in v . Since the word v was geodesic, the letters x and y were separated in v by a letter, say z , which does not commute with the elements x and $y = x^{-1}$. Hence z has to be cancelled out at some previous step, and, since x and y is the first pair of black letters to be cancelled, z is cancelled out by some red letter, say t . But t and z are separated in the original word uv by the letter x which is present in the word at all previous steps, preventing t and z from cancelling each other. This contradiction proves the lemma. \square

3. DIVISIBILITY THEORY

In this section we transfer the concept of divisibility from commutative algebra to the non-commutative setting of partially commutative groups. We follow the ideas of divisibility theory for the positive Artin semigroup [11]. However, we shall soon see that, in a simpler context of partially commutative groups, the whole group admits a good divisibility theory.

3.1. Divisibility. We use the notation $v \circ w$ for multiplication in the group \mathbb{G}_r when we wish to emphasise that there is no cancellation in the product of geodesic words representing the elements v and w , that is, $l(v \circ w) = l(v) + l(w)$. We shall use it also to denote concatenation of geodesic words (when the result is a geodesic word). We skip the symbol ‘ \circ ’ when it is clear from context that the product is cancellation free.

We start by introducing the notion of a *divisor* of an element; it will play a very important role in our paper.

Definition 3.1. Let u and w be elements in \mathbb{G}_r . We say that u is a left (right) *divisor* of w if there exists $v \in \mathbb{G}_r$ such that $w = u \circ v$ ($w = v \circ u$, respectively).

Notice that, in the context of free partially commutative monoids, divisors have been called *prefixes* [4].

We set $D_l(w)$ (correspondingly $D_r(w)$) to be the set of all left (correspondingly right) divisors of w . We shall also use the symbol $u \mid v$ to denote the left divisibility: $u \mid v$ is the same as $u \in D_l(v)$. The terms *common divisor* and *common multiple* of two elements are used in their natural meaning.

We shall work almost exclusively with the left divisibility, keeping in mind that the case of right divisibility is analogous. Indeed, the two concepts are interchangeable by taking the inverses of all elements involved.

Lemma 3.2. *Let v and w be fixed geodesic forms of two elements from \mathbb{G}_r . Then $v \mid w$ if and only if there are letters x_1, \dots, x_k in w such that*

- $x_1 \cdots x_k = u$.
- The letters x_1, \dots, x_k can be moved, by means of commutativity relations from R_X only, to the beginning of the word producing the decomposition

$$w = x_1 \cdots x_k \circ w'.$$

- This transformation does not change the relative position of other letters in the word, that is, if y_1 and y_2 are letters in w different from any of x_i , and y_1 precedes y_2 in w then y_1 precedes y_2 in w' .
- Moreover, if we colour letters x_1, \dots, x_k red and the remaining letters in w black, then a sequence of admissible transformations of w to $x_1 \cdots x_k \circ w'$ can be chosen in such way that first we made all swaps of letters of different colour and then all swaps of letters of the same colour.

Proof. This is a direct consequence of Transformation Lemma 2.3. \square

In view of Lemma 3.2, checking divisibility of elements amounts to simple manipulation with their geodesic forms.

Lemma 3.3. *The following two conditions are equivalent:*

- (a) $u \mid w$
- (b) $\lg(u^{-1}w) = \lg(w) - \lg(u)$.

Proof. By definition, $u \mid w$ means that $w = u \circ v$ for some v , that is, $\lg(w) = \lg(u) + \lg(v)$. But then, of course, $\lg(u^{-1}w) = \lg(u^{-1} \cdot u \cdot v) = \lg(v)$, proving (b).

Next, assume (b). We work with some fixed geodesic words representing u and w . For the equality $\lg(u^{-1}w) = \lg(w) - \lg(u)$ to be satisfied, $2 \lg(u)$ letters need to be cancelled in the word uw . By Lemma 2.6,

every cancellation involves a letter from u^{-1} and a letter from w , which means that u^{-1} is entirely cancelled.

Let us paint letters from u red, letters from w black, and repaint green those black letters which are to be cancelled by red letters. Obviously, if x is a green letter in w , then the letters in w to the left of it are either green or, if black, commute with x . Hence all green letters can be moved to the left, forming a word u' such that $w = u'v'$. Moreover, this transformation can be done in such a way that the relative order of green letters and the relative order of black letters are preserved. Now it is obvious that $u^{-1}u' = 1$ and $v' = u^{-1}w$. Hence $u' = u$ and $w = u \circ v'$, and, in other words, $u \mid w$. \square

Since the rewriting rules (2) and (3) give an algorithm for computing normal forms of elements, Lemma 3.3 yields an obvious algorithm which decides, for given elements u and w , whether $u \mid w$.

3.2. Abelian divisors and chain decomposition. We call an element $u \in \mathbb{G}_r$ *abelian* if all symbols in $\alpha(u)$ commute.

Lemma 3.4. *Let $w \in \mathbb{G}$. Then there exists the greatest (left) abelian divisor u of w in the sense that if v is any other abelian left divisor of w then $v \mid u$.*

We shall denote the greatest left abelian divisor of w by $\text{ad}(w)$.

Proof. $\text{ad}(w)$ is composed of all letters in a geodesic form of w which can be moved to the leftmost position (hence commute with each other). Notice that $\text{ad}(w)$ does not depend on the choice of a geodesic form of w . \square

Lemma 3.4 gives us a useful decomposition of elements in \mathbb{G}_r . Set $w = w_1$ and decompose

$$\begin{aligned} w_1 &= \text{ad}(w_1) \circ w_2 \\ w_2 &= \text{ad}(w_2) \circ w_3 \\ &\vdots \end{aligned}$$

Iteration of this procedure gives us a decomposition of w :

$$(5) \quad w = c_1 \circ c_2 \circ \cdots \circ c_t, \text{ here } c_i = \text{ad}(w_i).$$

The elements c_i , $i = 1, \dots, t$ are called *chains* of w , and decomposition (5) *chain decomposition*.

Algorithm 3.5 (Chain Decomposition).

INPUT: a word w of length n in a geodesic form.

OUTPUT: Abelian words c_1, \dots, c_n such that

$$w = c_1 \circ \dots \circ c_n$$

is the chain decomposition of w .

1° INITIALISE

1.1. Words $c_1 = 1, \dots, c_n = 1$ (chains).

1.2. Integers $m[x] = 0$ for $x \in X$ (chain counters).

2° FOR $i = 1, \dots, n$ DO

2.1. Read the letter $w[i]$ and its symbol x , $\{x\} = \alpha(w[i])$.

2.2. Make the set of symbols $Y \subseteq X$ which do not commute with x .

2.3. Compute the chain number for $w[i]$:

$$m[x] = 1 + \max_{y \in Y} m[y].$$

2.4. Append the chain:

$$c_{m[x]} \leftarrow c_{m[x]} \circ w[i].$$

END

Notice that this algorithm works in time $O(rn)$.

Example 3.6. The chain decomposition of the element $x_2x_5x_1x_3^{-1}x_1x_5x_4$ in the group

$$G = \langle x_1, \dots, x_5 \mid x_ix_j = x_jx_i \text{ for } |i - j| > 1 \rangle$$

is

$$x_2x_5x_5 \cdot x_1x_3^{-1}x_1 \cdot x_4.$$

This can be seen from the following table which captures the working of Algorithm 3.5. Here, the integers under the letters are current values of counters $m[x]$.

x_2	x_5	x_1	x_3^{-1}	x_1	x_5	x_4
1	1	2	2	2	1	3

3.3. The greatest common divisor and the least common multiple. Obviously, the relation ' $u \mid w$ ' is a partial ordering of $D_l(w)$. We shall soon see that $D_l(w)$ is a lattice. (This result is well known in the case of free partially commutative monoids [4, p. 150].) This will allow us to use the terms *greatest common divisor* $\text{gd}(u, v)$ and *least common multiple* in their usual meaning when restricted to $D_l(w)$. Without this restriction, the least common multiple of two elements u and v does not necessarily exist; consider, for example, elements $u = x_1$ and $v = x_2$ in the free group $F_2 = \langle x_1, x_2 \rangle$. Notice, however, that the greatest common divisor exists for any pair of elements in any right angled group.

To see that, consider geodesic forms of elements u and v ; the greatest common divisor of u and v is formed by all letters in v which are cancelled in the product $u^{-1}v$, see Lemma 2.7.

The structure of the partially ordered set $D_l(w)$ in the case of free abelian groups is quite obvious:

Example 3.7. Let $\mathbb{G}_r = \mathbb{Z}^r$ be the free abelian group of rank r . Assume $w = x_1^{\alpha_1} \cdots x_r^{\alpha_r}$ written in normal form. Then every divisor of w has the form $x_1^{\beta_1} \cdots x_r^{\beta_r}$, where $\text{sign}(\beta_i) = \text{sign}(\alpha_i)$ and $|\beta_i| \leq |\alpha_i|$, $i = 1, \dots, r$. In that special case, the set $D_l(w)$ is obviously a lattice. Indeed, if $u = x_1^{\beta_1} \cdots x_r^{\beta_r}$ and $v = x_1^{\gamma_1} \cdots x_r^{\gamma_r}$ are elements in $D_l(w)$ represented by their normal forms then

$$\begin{aligned} \text{gd}(u, v) &= x_1^{\phi_1} \cdots x_r^{\phi_r} \\ \phi_j &= \min\{|\beta_j|, |\gamma_j|\}, \quad \text{sign}(\phi_i) = \text{sign}(\alpha_i) \\ \text{lm}(u, v) &= x_1^{\psi_1} \cdots x_r^{\psi_r}, \\ \psi_j &= \max\{|\beta_j|, |\gamma_j|\}, \quad \text{sign}(\psi_i) = \text{sign}(\alpha_i) \end{aligned}$$

Lemma 3.8. Assume that $u \mid v$ and let $u = u_1 \cdots u_k$ and $v = v_1 \cdots v_l$ be chain decompositions of u and v . Then $k \leq l$ and $u_i \mid v_i$ for all $i \leq k$.

Proof. Since u_1 is an abelian divisor of v and v_1 is the greatest abelian divisor of v , we see that $u_1 \mid v_1$.

Next decompose $u = u_1 \circ u'$ and $v = v_1 \circ v'$. Notice that none of the letters of u' appear in v_1 , for otherwise that would have to be included in u_1 . Hence $u' \mid v'$ and we can conclude the proof by induction on $\text{lg}(u)$. \square

We can reverse the statement of Lemma 3.8 in the important case when u and v have a common multiple.

Lemma 3.9. Assume that $u \mid w$ and $v \mid w$. Take the chain decompositions $u = u_1 \cdots u_k$ and $v = v_1 \cdots v_l$ of u and v . Assume that $u_i \mid v_i$ for all $i \leq k$. Then $u \mid v$.

Proof. Let $w = w_1 \cdots w_t$ be the chain decomposition of w . By Lemma 3.8, $u_1 \mid w_1$ and $v_1 \mid w_1$. Denote $u' = u_2 \cdots u_k$, $v' = v_2 \cdots v_l$ and $w' = w_2 \cdots w_t$. Arguing as in the proof of Lemma 3.8, we see that no letter of u' or v' appears in w_1 and $u' \mid w'$ and $v' \mid w'$. We can conclude by induction that $u' \mid v'$ and that letters of u' can be chosen to lie in v' . Since u_1, v_1, w_1 are abelian divisors, we can conclude without loss of generality that letters of u_1 lies in v_1 . Now it is obvious that all letters of u can be chosen to belong to v and that $u \mid v$. \square

Proposition 3.10. $D_l(w)$ is a lattice with respect to the divisibility relation $|$.

Proof. This immediately follows from Lemma 3.8 and Lemma 3.9. \square

Lemma 3.11. Assume that $u | w$ and $v | w$. Take the chain decompositions $u = u_1 \cdots u_k$ and $v = v_1 \cdots v_l$ of u and v . Assume that $k \leq l$. If $k < l$, set $u_{k+1} = \cdots = u_l = 1$. In this notation,

$$\text{gd}(u, v) = \text{gd}(u_1, v_1) \cdots \text{gd}(u_k, v_k)$$

and

$$\text{lm}(u, v) = \text{lm}(u_1, v_1) \cdots \text{lm}(u_l, v_l).$$

Proof. This also immediately follows from Lemma 3.8 and Lemma 3.9. \square

3.4. Parabolic divisors. We set $D_{l,Y}(v)$ be the set of all left divisors of v from $\mathbb{G}(Y)$; we shall call them *parabolic divisors*.

Proposition 3.12. For an arbitrary element $v \in \mathbb{G}_r$ and subset $Y \subseteq X$ the set $D_{l,Y}(v)$ is a lattice. In particular, $D_{l,Y}(v)$ possesses unique maximal element $\text{gd}_Y(v)$.

Proof. This is an immediate corollary of Proposition 3.10. \square

3.5. Divisibility: further properties. We record an easy cancellation property:

Lemma 3.13. Assume that $u | w$. If $u = u_1 \circ u_2$ and $w = u_1 \circ w_2$, then $u_2 | w_2$.

Proof. Rewrite the condition $u | w$ as $w = u \circ w_1$, then, obviously, the product $w = u_1 \circ u_2 \circ w_1$ is also cancellation free. But we are also given that $w = u_1 \circ w_2$. It follows that $u_2 \circ w_1 = w_2$, hence $u_2 | w_2$. \square

Lemma 3.14. Let $x_i, x_j \in X \cup X^{-1}$. If $x_i | w$ and $x_j | w$ then $x_i \neq x_j^{-1}$. If, in addition, $x_i \neq x_j$ then $x_i x_j = x_j x_i$ and $x_i x_j | w$.

Proof. We work with a fixed geodesic form of w . It easily follows from Lemma 3.2 that

$$w \simeq u' \circ x_i \circ v'$$

and

$$w \simeq u'' \circ x_j \circ v''$$

while x_i commutes with every letter in the interval u' and x_j commutes with every letter in the interval u'' . But one of the letters x_i, x_j precedes the other in w , therefore x_i and x_j commute and they both can be moved to the beginning of the word w using only the commutativity

relations. If $x_i = x_j^{-1}$ then they can be cancelled, which contradicts our assumption that w is in geodesic form. Hence $x_i x_j$ is a geodesic word and therefore $x_i x_j \mid w$. \square

Another useful lemma (an analogue of this lemma for positive Artin semigroups can be found in [11]) is

Lemma 3.15 (Reduction Lemma). *Let $v, w \in \mathbb{G}_r$ and $x_i, x_j \in X \cup X^{-1}$ satisfy $x_i \circ v = x_j \circ w$. Then $x_i x_j = x_j x_i$ and there exists $u \in \mathbb{G}_r$ such that*

$$(6) \quad v = x_j \circ u \text{ and } w = x_i \circ u.$$

Proof. Set $t = x_i \circ v = x_j \circ w$ then $x_i \mid t$ and $x_j \mid t$. Then $x_i x_j = x_j x_i$ by Lemma 3.14, and, moreover, $x_i x_j \mid t$. Hence $t = x_i x_j \circ u$ for some u . Hence $v = x_i^{-1} t = x_j \circ u$ and $w = x_j^{-1} t = x_i \circ u$. \square

Lemma 3.16. *Let x be a letter and assume that $x \mid w$ and $v \mid w$. If $x \nmid v$ then $[\alpha(v), x_i] = 1$.*

Proof. As usual, we work with a geodesic form for w . Since we have $w = x \circ w_1$, the word $x \circ w_1$ can be transformed into $v \circ w_2$ using only the commutativity relations. If the letter x is a part of v in the latter decomposition, we obviously have $x \mid v$. Hence the letter x is a part of w_2 and can be moved to the leftmost position over all the letters in v , which means that x commutes with every symbol in $\alpha(v)$. \square

Lemma 3.17. *Let $p \mid w$ and $q \mid w$ and assume that $\text{gd}(p, q) = 1$. Then $\alpha(p) \cap \alpha(q) = \emptyset$ and $[\alpha(p), \alpha(q)] = 1$.*

Proof. Let $p_1 \cdot p_2 \cdots p_s$ and $q_1 \cdot q_2 \cdots q_t$ be the chain decompositions of p and q . Let x and y be letters such that $x \mid p_1$ and $y \mid q_1$. Since $\text{gd}(p, q) = 1$ then $x \nmid q$ and $y \nmid p$. By Lemma 3.16, $[x, \alpha(q)] = 1$ and $[y, \alpha(p)] = 1$. Notice also that $x \neq y^{\pm 1}$. Hence $\alpha(x) \neq \alpha(y)$. Set $p = x \circ p'$ and $q = y \circ q'$. Notice that $xy \mid w$; if $w = xy \circ w'$ is the corresponding decomposition then $p' \mid w'$ and $q' \mid w'$. If $g = \text{gd}(p', q') \neq 1$, then, since every letter in g commutes with xy , we easily see that $g \mid p$ and $g \mid q$, contrary to the assumption that $\text{gd}(p, q) = 1$. By induction, $\alpha(p') \cap \alpha(q') = \emptyset$ and $[\alpha(p'), \alpha(q')] = 1$, and the lemma follows immediately. \square

Proposition 3.18. *Let $p \mid w$ and $q \mid w$; suppose $r = \text{gd}(p, q)$ and set $p = r \circ p'$, $q = r \circ q'$. Then $\alpha(p') \cap \alpha(q') = \emptyset$, $[\alpha(p'), \alpha(q')] = 1$ and $\text{lm}(p, q) = r \cdot p' \cdot q' = r \cdot q' \cdot p'$.*

Proof. It immediately follows from Lemma 3.17. \square

Definition 3.19. Let $p, u, v \in \mathbb{G}_r$. We say that p is a *left divisor* of u with respect to v if and only if $p \mid u$ and $[\alpha(p), \alpha(v)] = 1$.

Proposition 3.20. *There exists the greatest left divisor of u with respect to v .*

We shall denote it $p_v(u)$.

Proof. This is a special case of parabolic divisors, see Proposition 3.12. \square

4. NORMAL FORMS ARISING FROM HNN EXTENSIONS

In this section we shall develop a more efficient approach to normal forms on partially commutative groups. It is based on a presentation of \mathbb{G}_r as an HNN -extension of the group \mathbb{G}_{r-1} .

4.1. HNN -normal form. Let $w \in \mathbb{G}_r$. We define the *HNN -normal form* $n(w)$ by induction on r . If $r = 1$ then \mathbb{G}_r is abelian and, by definition, the HNN -normal form of an element is its **SHORTLEX** minimal geodesic form. For the inductive step, we assume that $r > 1$ and, for $l < r$, set

$$X_l = X \setminus \{x_1, \dots, x_{l-1}\}.$$

Then for every group $\mathbb{G}_l = \langle X_l \rangle$ with $l < r$, the normal form is already defined. Then

$$\mathbb{G}_r = \langle \mathbb{G}_{r-1}, x_1 \mid \text{rel}(\mathbb{G}_{r-1}) \cup \{[x_1, x_j] = 1 \text{ for } [x_1, x_j] \in R_X\} \rangle$$

and the associated subgroup A of this HNN -extension is generated by all the symbols x_i for $i > 1$ which commute with x_1 . By the inductive assumption, the HNN -normal form $n(w)$ for a words w which does not contain the symbol x_1 is already defined.

It is a well-known fact in the theory of HNN -extensions [49] that every element of the group \mathbb{G}_r can be uniquely written in the form

$$(7) \quad w = s_0 x_1^{\alpha_1} s_1 x_1^{\alpha_2} s_2 \cdots s_{k-1} x_1^{\alpha_k} v,$$

where $s_0, \dots, s_{k-1}, v \in \mathbb{G}_{r-1}$ and s_i belong to a fixed system S of words such that the corresponding elements are left coset representatives of A in \mathbb{G}_{r-1} . The parameter k will be called the *syllable length* of w .

We choose the system of representatives S to satisfy the following two conditions.

- Each word $s \in S$ is written in the HNN -normal form in the group \mathbb{G}_{r-1} .
- If a letter $x_i \in A$ occurs in s then it can not be moved to the rightmost position by means of the commutativity relations of \mathbb{G}_{r-1} .

Clearly, such system of representatives exists. If we assume, in addition, that the word v is written in the HNN -normal form, then the word on the right hand side of (7) is uniquely defined for every element $w \in \mathbb{G}_r$; we shall take it for the HNN -normal form of w .

Proposition 4.1. *For every $w \in \mathbb{G}_r$, the SHORTLEX normal form w^* and the HNN -normal form $n(w)$ coincide.*

Proof. We use induction on r and show that every word written in the form w^* is written in the form $n(w)$. Let w^* be written in the form (7):

$$w^* \simeq u_0 x_1^{\beta_1} u_1 x_1^{\beta_2} u_2 \cdots u_l x_1^{\beta_l} u_{l+1}.$$

From the theory of HNN -extensions we extract that $k = l$, that $\alpha_i = \beta_i$ and that s_i and u_i are the elements of the same cosets of A in \mathbb{G}_{r-1} .

Now consider the word $n(w)$ and suppose that $n(w) \not\simeq w^*$. Then the word $n(w)$ is not SHORTLEX minimal in the class of all words representing the element w . Due to the Bokut-Shiao rewriting rules (Section 2.2.3) there exists an interval $n(w)[l, m]$ such that $n(w)[l] = x_i > x_j = n(w)[m]$ and the letter x_j commutes with every letter of this interval. Since, by the inductive assumption, $s_i^* \simeq n(s_i)$ and $v^* \simeq n(v)$, the interval $n(w)[l, m]$ can not be a subword of one of these words. Therefore $n(w)[l, m]$ involves x_1 . Notice that $x_i \neq x_1$, since $x_i > x_j$. But this implies that x_i commutes with x_1 , and thus $x_i \in A$. This contradicts the definition of the representatives $s_i \in S$, since x_i is involved in some representative s_p and can be taken to the rightmost position. \square

If we restrict our considerations to the free partially commutative monoid \mathbb{G}^+ of positive words in \mathbb{G} , then HNN -normal form of elements in \mathbb{G}^+ coincides with the *priority normal form* of Matiyasevich [22, 51].

4.1.1. *Cyclically reduced elements.* We say that $w \in \mathbb{G}_r$ is *cyclically reduced* if and only if

$$\lg(g^{-1}wg) \geq \lg(w)$$

for every $g \in \mathbb{G}_r$.

Observe that, obviously, for every element w there exists a cyclically reduced element conjugate to w .

One can find in the literature several slightly different definitions of cyclically reduced words and elements adapted for use in specific circumstances. In particular, a commonly used definition of cyclically reduced forms of elements of HNN -extension is given in [49]. We specialise it for the particular case

$$\mathbb{G}_r = \langle \mathbb{G}_{r-1}, x_1 \mid \text{rel}(\mathbb{G}_{r-1}), [x_1, A] = 1 \rangle.$$

Definition 4.2 (*HNN-cyclically reduced element*). An element

$$w = u_0 x_1^{\alpha_1} u_1 x_1^{\alpha_2} u_2 \cdots u_k x_1^{\alpha_k} u_{k+1}$$

is *HNN-cyclically reduced* if

- $u_0 = 1$ and all $u_i \notin A$, $i = 1, \dots, k$

and either

- $u_{k+1} = 1$ and $\text{sign}(\alpha_1) = \text{sign}(\alpha_k)$, or $u_{k+1} \notin A$.

5. CONJUGACY PROBLEM

Wrathal [62] found an efficient algorithm for solving the conjugacy problem in \mathbb{G} by reducing it to the conjugacy problem in the free partially commutative monoid $\mathbb{M} = M(X \cup X^{-1})$, which, in its turn, is solved in [48] by a reduction to pattern-matching questions (recall that two elements $u, v \in \mathbb{M}$ are *conjugate* if there exists $z \in \mathbb{M}$ such that $uz = zv$). In this section we shall give a *direct* solution to the conjugacy problem for partially commutative groups in terms of a conjugacy criterion for *HNN*-extensions and the divisibility theory developed in the preceding section. We also give solution to the *restricted conjugacy problem*: for two elements $u, v \in \mathbb{G}$ and a parabolic subgroup $\mathbb{G}(Y)$, decide whether there exists an element $z \in \mathbb{G}(Y)$ such that $z^{-1}uz = v$, and, if such elements z exist, find at least one of them.

5.1. Conjugacy Criterion for *HNN*-extensions. We begin by formulating a well-known result, Collins' Lemma [49]. It provides a conjugacy criterion for *HNN*-extensions.

Theorem 5.1 (Collins' Lemma). *Let $G = \langle H, t \mid t^{-1}At = B \rangle$ be an *HNN*-extension of the base group H with associated subgroups B and A . Let*

$$g = t^{\epsilon_1} h_1 \cdots h_{r-1} t^{\epsilon_r} h_r, \quad g' = t^{\eta_1} h'_1 \cdots h'_{s-1} t^{\eta_s} h'_s$$

*be normal forms of conjugate *HNN*-cyclically reduced elements from G . Then either*

- *both g and g' lie in the base group H , or*
- *neither of them lies in the base group, in which case $r = s$ and g' can be obtained from g by i -cyclically permuting it:*

$$t^{\epsilon_1} h_1 \cdots h_{r-1} t^{\epsilon_r} h_r \mapsto t^{\epsilon_i} h_i \cdots t^{\epsilon_r} h_r \cdot h_1 \cdots t^{\epsilon_{i-1}} h_{i-1}$$

and then conjugating by an element z from A , if $\epsilon_1 = 1$, or from B , if $\epsilon_1 = -1$.

In the case of partially commutative groups we shall use the following specialisation of Collins' Lemma.

Theorem 5.2 (Conjugacy criterion in terms of HNN -extensions). *Let*

$$G = \langle H, t \mid \{ t^{-1}at = a \mid a \in A \} \cup \text{rel}(H) \rangle$$

be an HNN -extension of the group H with the single associated subgroup A . Then each element of G is conjugate to an HNN -cyclically reduced element. There are two alternatives for a pair of conjugate cyclically reduced elements g and g' is from G :

- *If $g \in H$ then $g' \in H$ and g, g' are conjugated in base subgroup H .*
- *If $g \notin H$, suppose that*

$$g = t^{\epsilon_1} h_1 \cdots t^{\epsilon_r} h_r, \quad g' = t^{\eta_1} h'_1 \cdots t^{\eta_s} h'_s$$

are normal forms for g and g' , respectively. Then $r = s$ and g' can be obtained from g by cyclically permuting it and then conjugating by an element from A .

As a corollary of Theorem 5.2 we derive the following proposition.

Proposition 5.3. *Let Y be a subset of X . Then a pair of elements g and g' from $\mathbb{G}(Y)$ are conjugate in $G = \mathbb{G}(X)$ if and only if they are conjugate in $\mathbb{G}(Y)$.*

Proof. The group $\mathbb{G}(X)$ can be obtained from $\mathbb{G}(Y)$ as a multiple HNN -extension with letters from $X \setminus Y$. Therefore the statement follows from the first clause of the Conjugacy Criterion above.

Another proof follows from the observation that $\mathbb{G}(Y)$ is a retract of $\mathbb{G}(X)$, that is, there is an idempotent morphism $\pi : \mathbb{G}(X) \rightarrow \mathbb{G}(Y)$. If $h^{-1}gh = g'$ for some $h \in \mathbb{G}(X)$ then $\pi(h)^{-1}g\pi(h) = g'$ for $\pi(h) \in \mathbb{G}(Y)$. \square

Lemma 5.4. *Let $w \in G$ and $y \in X \cup X^{-1}$. Then either*

- *$y^{-1}wy = w$, that is, y commutes with every symbol in $\alpha(y)$, or*
- *one of the four alternatives holds:*
 - (1) *$y^{-1}wy = w$, i. e. y commutes with every letter involved in w ;*
 - (2) *$[w, y] \neq 1$, $y \in D_l(w)$, $y^{-1} \notin D_r(w)$ and $\lg(y^{-1}wy) = \lg(w)$;*
 - (3) *$[w, y] \neq 1$, $y \notin D_l(w)$, $y^{-1} \in D_r(w)$ and $\lg(y^{-1}wy) = \lg(w)$;*
 - (4) *$[w, y] \neq 1$, $y \in D_l(w)$, $y^{-1} \in D_r(w)$ and $\lg(y^{-1}wy) = \lg(w) - 2$;*
 - (5) *$[w, y] \neq 1$, $y \notin D_l(w)$, $y^{-1} \notin D_r(w)$ and $\lg(y^{-1}wy) = \lg(w) + 2$.*

Proof. The proof is obvious. \square

Let $\text{CR}(w)$ denote the set of all cyclically reduced elements conjugate to w .

Proposition 5.5. *Let w be a cyclically reduced element and take $v \in \text{CR}(w)$. Then*

- (a) $\text{lg}(v) = \text{lg}(w)$, and
- (b) v can be obtained from w by a sequence of conjugations by elements from $X \cup X^{-1}$:

$$w = v_0, v_1, \dots, v_k = v,$$

where each $v_i \in \text{CR}(w)$ and $v_i = y_i^{-1}v_{i-1}y_i$ for some $y_i \in X \cup X^{-1}$.

Proof. (a) is immediate from the definition of a cyclically reduced element.

For a proof of (b), assume the contrary and take two distinct elements u and v in $\text{CR}(w)$ with the following properties:

- there is a geodesic word $y = x_1 \cdots x_l$, $x_i \in X \cup X^{-1}$, such that $y^{-1}uy = v$.
- If we denote $y_i = x_1 \cdots x_i$ and set $u_i = y_i^{-1}uy_i$, so that $u_{i+1} = x_{i+1}^{-1}u_i x_{i+1}$, $i = 0, 1, \dots, l-1$, then not all u_i belong to $\text{CR}(w)$.
- The word y is shortest possible subject to the above conditions.

As usual, we work with geodesic forms of u and v . The choice of u , v and Y implies, in particular, that if $u_i = u_{i+1}$ then we can remove the letter x_i from the word y , which is impossible because of the minimal choice of y . Therefore $u_i \neq u_{i+1}$ for all $i = 0, \dots, l-1$. Let us colour letters from y^{-1} and y red and letters from u black. Let i be the first index such that $\text{lg}(u_i) = \text{lg}(u_{i-1})$. By our choice of u and v , $i > 1$ and $\text{lg}(u_{i-1}) > \cdots > \text{lg}(u_1) > \text{lg}(u)$, and by Lemma 5.4 the word $u_{i-1} = y_{i-1}^{-1} \circ u \circ y_{i-1}$ is geodesic.

Now let us colour letters x_i^{-1} and x_i green and consider the product $u_i = x_i^{-1} \cdot y_{i-1}^{-1} \cdot u \cdot y_{i-1} \cdot x_i$. In the process of admissible transformation of this product into a geodesic form, cancellations only happen with pairs of letters of different colours (Lemma 2.7). Moreover, no cancellation between red and black letters is possible, hence at least one of green letters has to cancel with a red or black one.

By our choice of i , $\text{lg}(u_i) \leq \text{lg}(u_{i-1})$, therefore some cancellations happen. If there is a cancellation within words $x_i^{-1}y_{i-1}^{-1}$ and $y_{i-1}x_i$, then, due to the inverse symmetry of these words, a symmetric cancellation also takes place, and the resulting geodesic word has the form $u_i = y'^{-1}uy'$ for some geodesic word y' which is shorter than y ; this contradicts the minimal choice of y .

Assume now that there were a cancellation between a green letter and red letters from the other side of the word, say, between x_i on the right and y_{i-1}^{-1} on the left. But this means that x_i commutes with every black letter in u and every red letter in y_{i-1} . Hence x_i commutes with every letter in u_{i-1} and $u_i = u_{i-1}$, a contradiction.

Hence a green letter cancels out a black letter; this means that green and red letters commute and $u_i = y_{i-1}^{-1}x_i^{-1}ux_iy_{i-1}$ and $\lg(x_i^{-1}ux_i) \leq \lg(u)$. Since $u \in \text{CR}(w)$, this means that $x_i^{-1}ux_i \in \text{CR}(w)$ and, replacing u by $x_i^{-1}ux_i$, we producing a pair of elements which match our initial choice but have shorter conjugating element y . This contradiction completes the proof. \square

The notion of cyclically reduced element of partially commutative group can be reformulated in terms of divisibility theory.

Proposition 5.6. *For an element $w \in \mathbb{G}_r$, the following two conditions are equivalent:*

- (a) *w is cyclically reduced;*
- (b) *if $y \in X \cup X^{-1}$ is a left divisor of w , then y^{-1} is not a right divisor of w .*

Proof. (a) obviously implies (b), so we only need to prove the converse. Assume that w satisfies (b) but is not cyclically reduced. Therefore there exists $v \in \mathbb{G}_r$ such that $\lg(v^{-1}wv) < \lg(w)$. Choose v such that $\lg(v)$ is the minimal possible. Let $v = y_1 \cdots y_l$ be a geodesic form of v . Denote $w_0 = w$ and

$$w_k = y_k^{-1} \cdots y_1^{-1} \cdot w \cdot y_1 \cdots y_k$$

and take the first value of k such that $\lg(w_k) < \lg(w_{k-1})$. As in the previous proof, colour the letters in v^{-1} red and letters in w black. The minimal choice of v implies that in the product $v^{-1}wv$ there is no cancellation of letters of the same colour, and, since in the sequence of elements w_0, \dots, w_{k-1} the length does not decrease at any step, at least one letter from each pair $\{y_i^{-1}, y_i\}$, $i = 1, \dots, k-1$, is not cancelled in the product $w_{k-1} = y_{k-1}^{-1} \cdots y_1^{-1} \cdot w \cdot y_1 \cdots y_{k-1}$. Since the next element, $w_k = y_k^{-1}w_{k-1}y_k$, has smaller length than w_{k-1} , this means that both letters y_k^{-1} and y_k cancel some black letters from w . To achieve this cancellation, y_k^{-1} and y_k have to commute with all previous red letters $y_1^{\pm 1}, \dots, y_{k-1}^{\pm 1}$, and hence with the corresponding black letters from w which were possibly cancelled in $w_{k-1} = y_{k-1}^{-1} \cdots y_1^{-1} \cdot w \cdot y_1 \cdots y_{k-1}$. But this means that y_k^{-1} and y_k are cancelled out in the product $y_k^{-1}wy_k$, that is, y_k is a left divisor of w while y_k^{-1} is a right one. \square

Since the set $CR(w)$ is finite, Proposition 5.5 provides a solution to the conjugacy problem for partially commutative groups. However one can not state that this algorithm runs in polynomial time until one finds a polynomial bound for the cardinality of the set $CR(w)$.

However, the situation is not that obvious, since the size of $CR(w)$ can be exponential in terms of the rank r of the group \mathbb{G}_r . Consider, for example, the free abelian group A freely generated by x_1, \dots, x_{r-1} and take for \mathbb{G}_r the free product $A \star \langle x_r \rangle$. Take $w = x_1 \cdots x_{r-1} x_r$. If $I \sqcup J = \{1, \dots, r-1\}$ is any partition of the set $1, \dots, r-1$ into disjoint subsets $I = \{i_1, \dots, i_k\}$ and $J = \{j_1, \dots, j_l\}$, then any element $x_{i_1} \cdots x_{i_k} \cdot x_r \cdot x_{j_1} \cdots x_{j_l}$ belongs to $CR(w)$. This gives us 2^{r-1} elements in $CR(w)$.

However, the conjugacy problem in free products as one just mentioned is trivial, which shows that the size of $CR(w)$ is not necessarily a good indicator of its difficulty. In Section 6 we give a polynomial time algorithm for solving the conjugacy problem in partially commutative groups.

5.2. Block decomposition. For a partially commutative group \mathbb{G}_r consider its graph Γ . The vertex set V of Γ is a set of generators X of \mathbb{G}_r . There is an edge connecting x_i and x_j if and only if $[x_i, x_j] \neq 1$. The graph Γ is a union of its connected components. Assume I_1, \dots, I_k are sets of letters corresponding to the connected components of Γ . Then $\mathbb{G}_r = \mathbb{G}(I_1) \times \cdots \times \mathbb{G}(I_k)$ and the words that depend on letters from distinct components commute.

Consider $w \in \mathbb{G}$ and the set $\alpha(w)$. For this set, just as above, consider the graph $\Gamma(\alpha(w))$ (it will be a subgraph of Γ). For this graph can be either connected or not. If it is not, then we can split w into the product of commuting words $\{w^{(j)} | j \in J\}$, where $|J|$ is the number of connected components of $\Gamma(\alpha(w))$ and the word $w^{(j)}$ involves the letters from j -th connected compound. Clearly, $[w^{(j_1)}, w^{(j_2)}] = 1$.

We shall call an element $w \in \mathbb{G}$ a *block* if and only if the graph $\Gamma(w)$ is connected.

As we already know every element w admits a block decomposition:

$$(8) \quad w = w^{(1)} \cdot w^{(2)} \cdots w^{(t)},$$

here $w^{(i)}$, $i = 1, \dots, t$ are blocks.

Proposition 5.7.

- (1) *The element w of a partially commutative group is cyclically reduced if and only if each block involved in decomposition (8) is cyclically reduced.*

- (2) Let $w = w^{(1)} \cdot w^{(2)} \cdots w^{(t)}$ and $v = v^{(1)} \cdot v^{(2)} \cdots v^{(s)}$ be cyclically reduced elements decomposed into the product of blocks. Then v and w are conjugate if and only if $s = t$ and, after some certain index re-enumeration, $v^{(i)}$ is conjugate with $w^{(i)}$, $i = 1, \dots, t$.

Proof. The first statement is clear. To prove the second statement, assume that the blocks of v and w are conjugate. Applying Proposition 5.3 we obtain that the blocks are conjugated in correspondent connected components of $\Gamma(w)$ and $\Gamma(v)$. But this implies that v and w are conjugate.

Conversely, suppose that v and w are conjugate. According to Proposition 5.5 there exists a sequence of cyclically reduced elements $w = v_0, \dots, v_k = v$. Here $v_i \in \text{CR}(w)$ and v_{i+1}, v_i are conjugated by a letter from $X \cup X^{-1}$, and moreover $l(v_{i+1}) = l(v_i)$. Therefore the second statement follows under induction on the length of this sequence. \square

The proposition above allows us to reduce the proof of most statements for cyclically reduced words in partially commutative groups to the case when the considered word is a block.

5.3. Conjugacy with respect to a subgroup. Let $\mathbb{G}_r = \mathbb{G}(X)$ be a partially commutative group and let Y be a proper subset of X . We shall now draw our attention to the question

‘when are two elements of $\mathbb{G}(X)$ conjugate by an element from $\mathbb{G}(Y)$?’.

Assume that w is a cyclically reduced block element such that $\alpha(w) = X$. By $w^{\mathbb{G}(Y)}$ we shall denote the subset of elements from $\mathbb{G}(X)$ conjugate with w by an element from $\mathbb{G}(Y)$.

Our nearest goal is to provide an algorithm for solving the problem of ‘being an element of $w^{\mathbb{G}(Y)}$ ’.

Proposition 5.8. *Let be w a cyclically reduced block element such that $\alpha(w) = X$. Then there exists a unique element w_0 of $w^{\mathbb{G}(Y)}$ such that:*

- (1) w_0 is cyclically reduced;
- (2) $D_{l,Y}(w_0) = \{1\}$.

Proof. EXISTENCE. Let $r = \text{gd}_Y(w)$, $w_1 = r \cdot w'$ and set $w = r^{-1}wr = w' \cdot w_1$. Clearly, $D_{l,Y}(w') = \{1\}$. If $D_{l,Y}(w' \cdot r) = \{1\}$ then set $w_1 = w_0$.

Otherwise $r = p_1 r_1$ and $p_1 = p_{w'}(r) \neq 1$, $[\alpha(p_1), \alpha(w')] = 1$. Since $\alpha(w') \cup Y = X$ and since w is a block element, then $\alpha(p_1) \subsetneq Y$. Set $w_2 = p_1^{-1}w_1 p_1 = w' r_1 p_1$. If $D_{l,Y}(w_2) = 1$ then the process terminates. Otherwise proceed to the next iteration. Then $p_2 = p_{w' r_1}(p_1) \neq 1$ and $\alpha(p_2)$ is a proper subset of $\alpha(p_1)$. This means that the process

terminates in no more than $|Y|$ steps. Notice that the number of steps in this procedure depends on the cardinality of the alphabet X and does not depend on the length of the word w .

After iterating this procedure no more than $l(w) - 1$ times we obtain the required element.

UNIQUENESS. Suppose w and v are two elements of $w^{\mathbb{G}(Y)}$ satisfying conditions (1) and (2) of Proposition 5.8. Choose $u \in \mathbb{G}(Y)$ such that $v = u^{-1}wu$. If $v \neq w$ then $u \neq 1$. Consider geodesic forms of u^{-1} , w and u and colour their letters red, yellow and green, correspondingly. A geodesic form for v is the result of admissible permutations and cancellations in the product $u^{-1}wu$. If some red letter is not cancelled out in v then $D_{l,Y}(v) \neq \{1\}$. Since cancellations can take place only between letters of different colours (Lemma 2.7), this means that each red letter has to be cancelled out either by a yellow letter (but then $D_{l,Y}(w) \neq \{1\}$, or by a green letter. In the latter case red and green letters cancel each other, leaving the yellow letter intact. Since the original relative order of yellow letters can be restored by virtue of Lemma 2.5, this means that $v = w$. \square

Definition 5.9. Assume that an element w satisfies the conditions of Proposition 5.8. Then the unique element w_0 constructed in the proposition is called the *exhausted form* of w with respect to the parabolic subgroup $\mathbb{G}(Y)$ and is denoted $e_Y(w) = w_0$. We omit the subscript Y when it is clear from the context.

Remark 5.10. The proof of Proposition 5.8 provides us with an algorithm for computation of $e(w)$.

Algorithm 5.11. (Computation of $e(w)$)

- 1° Compute $w_1 = \text{gd}_Y(w)$.
- 2° Find the geodesic decomposition $w = w_1 w_2$.
- 3° Compute the greatest left divisor of w_1 with respect to w_2 :
 $w_3 = p_{w_1}(w_2)$
- 4° Find the geodesic decomposition $w_1 = w_3 \cdot w_4$.
- 5° Compute $w_5 = p_{w_2 \cdot w_4}(w_3)$
- 6° Find the geodesic decomposition $w_3 = w_5 \cdot w_6$.
- \vdots \dots

OUTPUT: The geodesic decomposition for $e(w)$ is

$$e(w) = w_2 \cdot w_4 \cdots w_{2k} \cdot w_{2k-1}, \quad 2k \leq |Y|.$$

Definition 5.12. Let $v, w \in \mathbb{G}$. We say that v is a *factor* of w if there exist $u_1, u_2 \in \mathbb{G}$ such that

$$w = u_1 \circ v \circ u_2.$$

Notice that if W is a cyclically reduced block element and $e(w)$ is its exhausted form then it follows from the algorithm for computation of $e(w)$ that $e(w)$ is a factor of w^r , where $r = |Y|$.

5.4. An algorithm for solving the Conjugacy Problem. The conjugacy criterion for HNN -extensions (Theorem 5.2), Propositions 5.5, 5.8 and Remark 5.10 provide an algorithm for deciding whether the elements $g, h \in \mathbb{G}_r$ are conjugate or not.

The description of procedures which are used in the steps of the scheme is given in Section 6.

Algorithm 5.13 (Conjugacy Problem).

INPUT: Two elements g and h .

OUTPUT: ‘Yes’ if g and h are conjugate.

- 1° Compute the normal forms g^* and h^* for elements g and h correspondingly.
- 2° Using the division algorithm and Proposition 5.6 compute cyclically reduced elements g' and h' of \mathbb{G}_r which are conjugate to g^* and h^* respectively.
- 3° If one of either g' or h' is an element of \mathbb{G}_{n-1} , while the other is not, then g and h are not conjugate. If both g' and h' lie in \mathbb{G}_{n-1} , then solve the conjugacy problem in a partially commutative group \mathbb{G}_{n-1} of lower rank.
- 4° Both g' and h' are not the elements of \mathbb{G}_{n-1} . Cyclically permuting a cyclically reduced element we may assume that its first letter is x_1^ϵ , i.e. we obtain cyclically reduced and HNN -cyclically reduced word simultaneously. Regarding g' and h' as words written in form (7) compare their syllable lengths k_1 and k_2 . If $k_1 \neq k_2$, then the elements g and h are not conjugate.
- 5° If $k_1 = k_2$, then compute the block decompositions $w^{(1)} \dots w^{(t)}$ and $v^{(1)} \dots v^{(p)}$ for g' and h' . If the number of factors in the block decompositions for g' and h' are distinct (that is, $t \neq p$), then, due to Proposition 5.7, the elements are not conjugate.
- 6° Construct the graphs $\Gamma(g')$ and $\Gamma(h')$. If the connected components of these graphs do not coincide, then, according to Proposition 5.7, some of the blocks $w^{(i)}$ and $v^{(i)}$ for g' and h' are not conjugate and so are g and h .
- 7° If $t = p > 1$ and $\alpha(w^{(i)}) = \alpha(v^{(i)})$ for some block decompositions of g' and h' , then the conjugacy problem for g and h is reduced to the conjugacy problem for pairs of words $\{w^{(i)}, v^{(i)}\}$, $i = 1, \dots, t$ in the groups $\mathbb{G}(\alpha(w^{(i)}))$ of lower ranks.

- 8° If $t = p = 1$ and $\alpha(g') \subset X$, $\alpha(g') \neq X$ then the conjugacy problem for g and h is reduced to the one in group $\mathbb{G}(\alpha(g'))$ of lower rank.
- 9° If $t = p = 1$ and $\alpha(g') = X$. Let w_i denote the i -cyclical permutation of the word

$$w = s_0 x_1^{\alpha_1} s_1 x_1^{\alpha_2} s_2 \cdots s_{k_2-1} x_1^{\alpha_{k_2}} v.$$

According to the conjugacy criterion, it suffices to solve the conjugacy problem for pairs of elements $\{w_i, h'\}$, $i = 0, \dots, k_2$ related to associated subgroup A . To complete Step 9° compute the elements $\{e(w^{(i)}), e(h')\}$, $i = 0, \dots, k_2$ (see the scheme for computation of an exhausted element in the end of the Section 5.3). If every pair $\{e(w^{(i)}), e(h')\}$ is a pair of different words, then g' and h' are not conjugate. If there exists an index i_0 such, that $e(w^{(i_0)}) = e(h')$, then the words g' and h' are conjugate in \mathbb{G}_r and so are g and h .

END

Remark 5.14. Let w be a geodesic block element. It follows from algorithm for conjugacy problem that there might be many exhausted elements conjugate to w . We consider the ShortLex-minimal among them and call it the *canonical representative* for the class $\{w^{\mathbb{G}_r}\}$

5.5. Double cosets of parabolic subgroups. Consider a partially commutative group $G = \langle X \mid R_X \rangle$ and two parabolic subgroups $H_1 = \mathbb{G}(Y)$ and $H_2 = \mathbb{G}(Z)$, $Y, Z \subset X$.

In this section, we outline an algorithm which decides, for given elements g_1 and g_2 of G , whether they belong to the same double coset $H_1 g H_2$ or not. And if they do, we shall show how one can easily find the elements h_1 from H_1 and h_2 from H_2 such that $g_1 = h_1 g_2 h_2$.

Definition 5.15. The element $g' \in H_1 g H_2$ is called a *canonical representative* of the double coset $H_1 g H_2$ if and only if $\text{gd}_{l,Y}(g') = \text{gd}_{r,Z}(g') = 1$.

Using induction on the length of an element (for the elements of $H_1 g H_2$) we show that in each coset $H_1 g H_2$ there exists a canonical representative. If the element g is canonical then the algorithm terminates. Otherwise, say, $1 \neq r = \text{gd}_{l,Y}(g)$ and $g = r \circ g_1$. Then $g_1 \in H_1 g H_2$ and $l(g_1) < l(g)$. Therefore the computation of the canonical representative for $H_1 g H_2$ is equivalent to “exhausting” the element g on the left and on the right (see Definition 5.9). We estimate the complexity of the algorithm for computation of $\text{gd}_{l,Y}(g)$ in Section 6.5.

Proposition 5.16. *Every double coset H_1vH_2 contains a unique canonical representative.*

Proof. Assume that v and w are two canonical representative of H_1vH_2 . We may assume without loss of generality that $v \neq 1$ and $w \neq 1$; we also assume that v and w are written in geodesic form. Then $H_1vH_2 \neq H_1H_2$. Assume that the length of v is lower or equal than the length of w . Then $w = h_1vh_2$, where h_1 and h_2 are written in geodesic form and $l(h_1) + l(h_2) = s$ is minimal. By our choice of h_1 and h_2 , no letter of h_1 (correspondingly, h_2) cancels with a letter of h_2 (correspondingly, h_1). Therefore at least one letter y from either h_1 or h_2 cancel a letter y^{-1} in the word v (see Section 6.2). This leads to a contradiction with the definition of a canonical representative. \square

6. COMPLEXITY OF ALGORITHMS: SOME ESTIMATES

6.1. Turing machines and the definition of complexity of algorithms. Let G be a finitely presented group,

$$(9) \quad G = \langle X \mid R \rangle.$$

The algorithms we construct in this paper are term rewriting systems. In particular, an algorithm \mathcal{A} takes a given word w in alphabet $X \cup X^{-1}$ as input and computes another word $v = \mathcal{A}(w)$. We adopt a version of Church's Thesis and assume that all our procedures are implemented on a multi-taped Turing machine M . We denote by $t_M(w)$ the run time of M computing an output word for a fixed initial input word w .

As usually, we introduce the *worst case complexity* of \mathcal{A} as an integer valued function

$$(10) \quad t_M(n) = \max \{ t_M(w) \mid |w| = n \}.$$

Here, $|w|$ denotes the length of a word w in the alphabet $X \cup X^{-1}$.

An algorithm \mathcal{A} is called *polynomial* if there exists its realization on Turing machine M such that $t_M(n)$ is bounded by a polynomial function of n .

The second definition of complexity is preferable in terms of practical, experimental use. We shall define '*average case complexity*' by the following timing function:

$$(11) \quad t_{M,C}(n) = \frac{\sum_{|w|=n} t_M(w)}{|S_n|},$$

here $|S_n|$ stands for the number of all words of length n .

The algorithm \mathcal{A} is called *polynomial on average* if and only if there exists its realization on Turing machine M such that $t_{M,C}(n)$ is bounded by a polynomial function of n .

It is natural and convenient to divide all algorithms into two classes, polynomial and non-polynomial. However, it is not sufficiently precise when we look at the practical applications of algorithms.

Consider, for example, a simple question. Given a letter x from an alphabet X and a word W in X , does w contain an occurrence of x ? When the inputs are represented as sequences of symbols on the tape of a Turing machine, this simple problem has linear time complexity in terms of the length of w . Even more counterintuitive is the complexity of the deletion of a letter from a word: we remove the letter and then have to close the gap, which involves moving a segment of the word one position to the left, rewriting it letter after letter. However, if the word is presented as collection of triples

(letter, pointer to the previous letter, pointer to the next
letter),

then deletion of a letter requires a modification of at most three triples and can be done in constant time.

This simple example shows that assertions that certain algorithms works in linear, quadratic, etc., time, are relative and depend on the data formats used.

This is one of the reasons why computer scientists use a much more pragmatic setup for the complexity of algorithms, using more natural data formats and models of computation (for example, random access machines) [17]. We use [17] as the main source of references to standard results about complexity of algorithms used in this paper, and, in particular, to various modifications and generalizations of sorting algorithms. The latter are indispensable building blocks for our word processing algorithms. The only modification we introduce is that in [17] the relations between members of sequences of letters (numbers) remain constant, while in our case they are changing in the process of work of our higher level word processing algorithms. To emphasize the difference, we shall refer to our algorithms as *dynamic* sorting algorithms.

6.2. Conventions. We want to make our results independent of varying and diverse definitions of complexity of algorithms. To that end we choose the following tactic.

First of all, we select certain operations on words which we regard as elementary. With a single exception, all these operations are well

known in the theory of algorithms and we use classical results about their complexity.

Secondly, we understand complexity of an algorithm \mathcal{A} as the (worst possible) number of elementary operations it is using when working on inputs of length n and denote it $f_{\mathcal{A}}(n)$. We formulate our results in the standard O -notation: $O(f_{\mathcal{A}}(n)) = g(n)$ if there exists positive constant c and integer n_0 such that

$$0 \leq f_{\mathcal{A}}(n) \leq cg(n)$$

for all $n > n_0$.

To introduce elementary operations, we need some notation.

Let w be a word in alphabet $X \cup X^{-1}$. Denote by $l(w)$ the standard length of w , that is, the number of letters it contains. We denote as $w[i]$ the letter in the i -th position of the word w , while $w[i, j]$ denotes the subword $w[i]w[i+1] \cdots w[j]$.

Now we list our elementary operations. First three of them will be also called *basic* operation.

6.2.1. *Cancellation.* If $w[j] = w^{-1}[j+1]$, then, after deleting letters $w[j]$ and $w[j+1]$ from w , we have a word w' with $l(w') = l(w) - 2$. The complexity of this operation is a constant [17].

6.2.2. *Transposition of a letter and an admissible interval.* We shall use two types of *admissible* intervals in a geodesic word w . If a letter $x = w[j]$ commutes with all letters in the interval $w[j+1, i]$, we say that the interval $w[j+1, i]$ *admits* swapping with x . In that case, the result of the transposition applied to the word

$$\cdots w[j]w[j+1, i] \cdots$$

is the word

$$w' = \cdots w[j+1, i]w[j] \cdots$$

Another type of admissible intervals is the following: if w is a geodesic word and $w = xw'$, then w' is an admissible interval. In that case we can swap w' and x .

If one uses pointers, the transposition of a letter and an admissible interval requires only linear time (and constant time if the length of the interval is bounded from above by some constant K).

We described the left hand side version of the operation. Of course, the swap can be made the other way round.

6.2.3. *Query.* Let x be a fixed letter of our alphabet X and $w = y_1 \cdots y_l$ a word, $y_i \in X \cup X^{-1}$. Then we set

$$\ln(w) = \begin{cases} y_{i_0} & \text{if } [y_{i_0}, x] = 1 \text{ and } [y_i, x] \neq 1 \text{ for all } i < j_0, \\ 1 & \text{if such } i_0 \text{ does not exist.} \end{cases}$$

The right hand side version of this operation is denoted $\text{rn}(w)$.

6.2.4. *Abelian sorting.* Assume that all letters in the interval $w[i, j]$ commute pairwise (we call such intervals *abelian*). We apply to the interval one of the sorting algorithms of [17, Chapter 2] and rewrite it in the increasing order (with respect to indices).

Most algorithms in [17, Chapter 2] have worst case run time $O(n^2)$ and average time complexity $O(n \ln n)$. The latter is also a lower bound for comparison algorithms [17, Section 9.1]. We note that there is a sorting algorithm working in linear time [17, Section 9].

6.2.5. *Computation of the maximal left divisor which commutes with the letter x .* Given a word w and letter x , we want to find $p = p_x(w)$, the maximal left divisor of w which commutes with x , and rewrite the word w in the form pw' without changing the length of the word. We shall show in Section 6.5, that this operation can be performed in linear time.

Again, we can similarly introduce the right hand side version of this operation.

6.2.6. *Iterated divisors.* Let $Z = \{z_1, \dots, z_k\}$ be a linearly ordered (in a way possibly different from ordering by indexes) subset of the alphabet X . Define by induction

$$p_{\{z_1, \dots, z_i\}}(w) = p_{z_i}(p_{\{z_1, \dots, z_{i-1}\}}(w))$$

and

$$p_Z(w) = p_{\{z_1, \dots, z_k\}}(w).$$

This is an iterated version of the previous operation: finding the maximal left divisor commuting with a given letter.

6.2.7. *The greatest common divisor and the least common multiple of two geodesic words in the free abelian group.* The corresponding algorithm is relatively simple and clearly explained in Example 3.7. It can be performed in linear time.

6.2.8. *Divisors in abelian group.* Let $Y \subseteq X$. We need to compute $\text{gd}_Y(w)$ in the special case when \mathbb{G}_r is an abelian group. This can be done in linear time.

6.2.9. *Cyclic permutation of a geodesic word.* This operation transforms a geodesic word $w = p \cdot q$ into $x' = q \cdot p$.

Complexity of this operation depends on the model of computation. It is linear in terms of the basic operation 6.2.2. However, if we represent the word as a sequence of pointers, then the complexity is constant.

6.2.10. *Occurrences of letters.* Given a word w , $\alpha(w)$ denotes the set of letters occurring in w . If pointers are used, then the set $\alpha(w)$ and the intersection of two such sets $\alpha(u) \cap \alpha(v)$ can be found in constant time.

6.2.11. *Connected components of the graph $\Gamma(w)$.* It is well known that connected components of a graph with v vertices and e edges can be found in linear time $O(v+e)$ [17, §23]. In our context $v = r$ and $e \leq r^2$, which gives a quadratic bound in terms of r .

6.3. Complexity of the algorithm for computation of normal forms. Let \mathbb{G}_r be a partially commutative group of rank r . If $r = 1$ then \mathbb{G}_r is the infinite cyclic group generated by element x_1 . In order to reduce to a normal form a word w of length n in alphabet $\{x_1, x_1^{-1}\}$, one needs at most $\lceil n/2 \rceil$ cancellations.

Let now $r > 1$. Our alphabet $X = \{x_1, \dots, x_r\}$ is linearly ordered as

$$x_1 < \dots < x_r.$$

Then \mathbb{G}_r is the *HNN*-extension of the group $\mathbb{G}_{r-1} = \langle x_2, \dots, x_r \rangle$ by the stable letter x_1 with the associated subgroup

$$A = \langle x_i \mid [x_i, x_1] = 1, i \geq 2 \rangle.$$

The group \mathbb{G}_r can be written in generators and relations as

$$\mathbb{G}_r = \langle \mathbb{G}_{r-1}, x_1 \mid \text{rel } \mathbb{G}_{r-1}, x_1^{-1} a x_1 = a \text{ for all } a \in A \rangle.$$

We shall use a standard algorithm for computing normal forms of elements of *HNN*-extensions, adapted to our concrete situation. We shall denote it \mathcal{N} . Given a word w of length n in alphabet $X \cup X^{-1}$, it is converted to normal form $N(w)$. Let $f_r(n)$ be the number of elementary operations needed for computation of a normal form of arbitrary word of length n , and $g_r(n)$ the same quantity, but restricted to geodesic words of length n .

Proposition 6.1. *In this notation, $f_r(n) \leq n^2$, while $g_r(n) \leq 2n$.*

Proof. If $r = 1$, then these estimates easily follow from the estimates for infinite cyclic group given at the beginning of this section.

Let now $r > 1$ and assume, by way of induction on n , that for all $i = 1, 2, \dots, r - 1$ the bounds

$$f_i(n) \leq n^2 \text{ and } g_i(n) \leq 2n$$

are true. Consider first the function g_r . By Lemma 2.3, the transformation of a geodesic word to normal form does not change its length, that is, cancellations are not used.

Let $w = w' \cdot x$ with $x \in X \cup X^{-1}$. Then $n(w) = n(n(w') \cdot x)$. By the inductive assumption, the transformation of w' into $n(w')$ requires at most $2(n - 1)$ elementary operations. To prove the desired bound for g_r , we have to show that transformation of $n(w') \cdot x$ into $n(w)$ requires at most two elementary transformation, which can be show by easy case-by-case considerations.

The proof of a bound for $f_r(n)$ can be done similarly. the only difference is that we encounter the so-call *pinch*, that is, the situation when $n(w') = \dots vx_1^\epsilon u$, $u \in A$, $x = x_1^{-1}$ and $v \in \mathbb{G}_{r-1}$. In that case $n(w')x = \dots vx_1^\epsilon ux_1^{-\epsilon}$ and we need to apply the following elementary operations:

- we have to swap u and $x_1^{-\epsilon}$,
- cancel $x_1^\epsilon x_1^{-\epsilon}$,
- apply some number of elementary operations for converting vu to normal form.

The word vu is geodesic and does not belong to A , for otherwise $n(w')$ is not a normal form. Indeed, if the word vu is not geodesic, then, according to the Cancellation Property (Lemma 2.2), there exists an interval $x_k vu[i + 1, j]x_k^{-1}$ and letter x_k which commutes with all letters in the interval $vu[i + 1, j]$, and, by our assumption, with the letter x_1 . Applying again Lemma 2.2, we conclude that the word $n(w')$ is not geodesic, a contradiction. We see now that need at most $2(n - 2)$ elementary operations for converting vu to normal form. The total number of operations now is bounded by

$$(n - 1)^2 + 2n - 4 < n^2,$$

which completes the proof. \square

6.4. Complexity of the algorithm for computation of the chain decomposition. Assume that we have a geodesic word w . Recall (see Section 3) that

$$w = w_1 \cdots w_l$$

is the product of chains and w_1 is the largest abelian divisor of w , and if we write $w = w_1 \circ w'$ then w_2 is the largest abelian divisor of w' , etc.

The largest abelian divisor of a geodesic word w is computed by the following procedure. Let z_1 be the first letter of w and $p_1 = p_{z_1}(w)$ the result of application of the elementary operation of type (iv). Set $w = p_1 w_1$. Applying a sorting algorithm to the word p_1 , we can rewrite $p_1 = z_1^{\alpha_1} \circ p'_2$, where letter z_1 does not occur in p'_2 . Let z_2 be the first letter of p'_2 . Then $p_2 = z_1^{\alpha_1} \cdot p_{z_2}(p'_2) = z_1^{\alpha_1} z_2^{\alpha_2} p'_3$. This process continues no more than r times and requires at most $3r$ elementary operations. The procedure produces an abelian divisor $w_1 = z_1^{\alpha_1} \cdots z_k^{\alpha_k}$ of and a geodesic decomposition $w = w_1 \cdot w'$. It is easy to see that w_1 is the largest left abelian divisor of w in the sense that any other left abelian divisor divides w_1 .

Denote by $h_r(n)$ the number of elementary operations needed (in the worst case scenario) for computation of the chain decomposition of a geodesic word of length n .

Proposition 6.2. $h_r(n) \leq 3rn$.

Proof. The proof immediately follows from the description of the procedure. \square

6.5. Complexity of algorithms for computation of divisors. Let u and v be left divisors of an element w . We assume that u and v are written in geodesic form. We compute, with the help of the algorithm for computation of a the chain decomposition, the first chains u_1 and v_1 in the chain decompositions of elements u and v :

$$u = u_1 \cdot u', \quad v = v_1 \cdot v'.$$

We compute next the set

$$\alpha(u_1) \cap \alpha(v_1) = \{z_1, \dots, z_k\}.$$

If this set is empty then $\text{gd}(u, v) = 1$. Otherwise, using a sorting algorithm and algorithm 6.2.8, we compute

$$\text{gd}(u_1, v_1), \quad \text{lm}(u_1, v_1),$$

move to the pair of elements u', v' and make the same calculations with the second chain (see Section 3 for details). If the length of words u and v is bounded from above by n , we have to repeat these procedures no more than n times. Denote these procedures \mathcal{GD} and \mathcal{LM} and their complexity functions $\text{GD}(n)$ and $\text{LM}(n)$. Since every step of these algorithms requires 4 elementary operation, we established the following fact.

Proposition 6.3.

$$\text{GD}(n) \leq 4n, \quad \text{LN}(n) \leq 4n.$$

Let $Y \subset X$ be a proper subset of X . Given a geodesic word w , denote by $\text{gd}_Y(w)$ the maximal left divisor of w subject to the condition that all its letters belong to Y . Let GD_Y denotes the time complexity of the algorithm for finding $\text{gd}_Y(w)$.

This is how the algorithm works. First we compute, by means of the algorithm for chain decomposition, the first chain w_1 of w and the decomposition $w = w_1 \cdot w'$. According to Section 6.4, we need for that at most $3r$ elementary operations. Next we find the maximal Y -divisor u_1 of the abelian word w_1 , as defined in Proposition 3.12, with the help of elementary operation 6.2.8. Similarly, we find maximal Y -divisors u_2, u_3, \dots of w_2, w_3, \dots . The desired divisor is $u_1 u_2 \cdots u_k$.

If we denote, as usual, $n = l(w)$, then we come to the following result.

Proposition 6.4.

$$\text{GD}_Y(n) \leq (3r + 1)n.$$

This also gives us the complexity of the algorithm $\mathcal{P}_{x,Y}$ which finds the left divisor $p = p_x(w)$ of the word w maximal subject to condition that all letters in p commute with x .

Proposition 6.5.

$$P_{x,Y}(n) \leq (3|Y| + 1)n.$$

6.6. The complexity of computing cyclically reduced forms.

The algorithm \mathcal{CR} takes as input a geodesic word w and outputs a cyclically reduced word $\text{cr}(w)$ which is conjugate to w and is produced as follows. Denote by \overleftarrow{w} the reverse of the word w (that is, it is written by the same letters in the reverse order). Let w_1 and \overleftarrow{w}_1 be the first chains in the chain decomposition of words w and \overleftarrow{w} , correspondingly. Compute

$$\alpha(w_1) \cap \alpha(\overleftarrow{w}_1) = \{z_1, \dots, z_{k_1}\},$$

which is elementary operation 6.2.10. The total number of elementary operations in this step of the algorithm is at most $6r + 1$. If the intersection is empty, then $\text{cr}(w) = w$ and the algorithm stops. Otherwise we apply k_1 operations of cyclic reduction (see the fourth clause of Lemma 5.4), transforming w to a geodesic word w' . Then we repeat the same procedure for w' , etc. Obviously, this process requires at most $\lceil n/2 \rceil$ steps. Collecting the time complexities of all stages of reduction, we see that

Proposition 6.6.

$$\text{CR}(n) \leq \frac{3r}{2}n^2 + O(n).$$

6.7. The complexity of block decomposition and finding the exhausted form. Denote by \mathcal{B} the algorithm of block decomposition which rewrite a geodesic word w as the product of blocks:

$$w = w^{(1)}w^{(2)} \dots w^{(k)}.$$

The algorithm works as follows.

Algorithm 6.7.

- 1° Find connected components C_1, \dots, C_k of the graph $\Gamma(w)$ (this is an elementary operation 6.2.11).
- 2° Initialise $w^{(1)} = \emptyset, \dots, w^{(k)} = \emptyset$.
- 3° For $i = 1, \dots, k$ do
 - If $w[i] \in C_j$ then $w^{(j)} \leftarrow w^{(j)} \circ w[i]$.
- 4° Return $w^{(1)}, \dots, w^{(k)}$.

Proposition 6.8. $B(n) \leq O(r^2) + O(n)$.

Now we want to estimate the complexity of the algorithm for finding the exhausted form of a cyclically reduced word w . We impose on w the extra condition that w is a block and also $\alpha(w) = X$.

This algorithm \mathcal{E}_Y was described in details in the proof of Proposition 5.8. The number of elementary operations it requires does not depend on the length of w and is bounded above by a constant $|Y|$, $Y \subset X$.

Proposition 6.9. $E_Y(n) \leq |Y|$.

6.8. The complexity of the algorithm for the conjugacy problem. In our usual notation, let u and v be two words in the alphabet X . The conjugacy algorithm \mathcal{C} , when applied to words u and v , decides, whether the elements of $G = \langle X \rangle$ represented by the words u and v conjugate or not. The algorithm was described in Section 5.4, here we only discuss its complexity. We shall prove two closely related bounds for its complexity. In one case, we estimate the function $C(n, r)$ which gives the number of elementary operations 6.2.1–6.2.11 needed for deciding, in the worst case scenario, the conjugacy of two words of length at most n .

Our second estimate is motivated by the following observation. The majority of our algorithms are algorithms of dynamic sorting, based on comparing pairs of letters. Therefore for the second estimate we limit the list of elementary operations to just three: cancellation (6.2.1), transposition of a letter and an admissible interval (6.2.2) and query (6.2.3). All other elementary operations have linear complexity with respect to these three basic operations. This observation follows from the simple fact that, in the free abelian group, all our algorithms work in

linear time modulo these three basic operation. Therefore it is natural to produce an estimate of the complexity of the conjugacy algorithm \mathcal{C} in terms of the three basic operations.

6.8.1. *The first estimate.*

Proposition 6.10. $C(n, r) \leq O(r^2) \cdot O(n^2)$.

Proof. The algorithm starts with reducing the input elements u and v to normal form $n(u)$ and $n(v)$, which requires quadratic time in the worst case. We do not repeat this operation at later steps of the algorithm. Then we have to find block decompositions of the words $n(u)$ and $n(v)$, which can be done in linear time. If the number of blocks is at least two, then the algorithm either gives the negative answer, or reduces the problem to groups of smaller rank. By induction, we can assume that the normal words $u = n(u)$ and $v = n(v)$ each has only one block. Now the algorithm finds cyclically reduced forms $cr(u)$ and $cr(v)$. These operation have to be called just once, and they can be done in quadratic time at worst. We replace $u \leftarrow cr(u)$ and $v \leftarrow cr(v)$. If one of $\alpha(u)$ or $\alpha(v)$ is a proper subset of X , then the algorithm either gives negative answer, or reduces the problem to a group of smaller rank. Therefore we assume that $\alpha(u) = \alpha(v) = X$ and rewrite u and v in normal form with respect to the letter x_1 . This can be done in linear time modulo the timing of elementary operations. After that the algorithm computes at most n i -cyclic permutations of elements in u , producing words u_1, \dots, u_k , $k < n$, and computes exhausted forms $e(u_1), \dots, e(u_k), e(v)$, which requires at most linear number of operations in total, since every exhausted form can be found in constant number of elementary operations. Finally, we have to check k equalities $e(u_1) = e(v), \dots, e(u_k) = e(v)$, which again requires quadratic number of elementary operations. The elements u and v are conjugate if and only if $e(u_i) = e(v)$ for some i . \square

6.8.2. *The second estimate.* Let $\bar{C}(n, r)$ be the number of basic operations (6.2.1), (6.2.2) and (6.2.3) needed for executing the conjugacy algorithm on words of length at most n . Since elementary operations can be reduced to linearly bounded number of basic operations, we come to the following result.

Theorem 6.11. $\bar{C}(n, r) \leq O(r^2 n^3)$.

6.9. Complexity of the conjugation problem restricted to cyclically reduced words. It is easy to observe that, of the algorithms introduced in the present paper, the most complicated from the point of view of the worst-case complexity are the following:

- the algorithm for computation of the normal form, and
- the algorithm for computation of the cyclically reduced form.

As a rule, higher level algorithms (for solving the conjugacy problem, for instance) require very few applications of these complicated algorithms for almost all input words. We expect, by analogy with the work [9], that the average time complexity of algorithms for computation of normal and cyclically reduced forms is much lower than their worst case time complexity.

Therefore it makes sense to record the estimates for the complexity of the algorithm for conjugacy problem with inputs restricted to cyclically reduced words. Analysis of the proofs of Proposition 6.10 and Theorem 6.11 easily leads to the following result.

Theorem 6.12. *When restricted to cyclically reduced input words, the algorithm \mathcal{C} for conjugacy problem requires at most $O(rn)$ elementary operations or $O(rn^2)$ basic operations.*

Acknowledgements. The authors are grateful to Alexandre Borovik whose work on the paper far exceeded the usual editor's duties.

7. APPENDIX

Nicholas Touikan has drawn our attention to the fact that the proof of Proposition 4.1 is incorrect. This proposition, however, is never used in proofs of any other results of the paper and was written for the sake of completeness of the exposition. Furthermore, Proposition 4.1 is correct (including its proof) if instead of the ShortLex ordering on the free group $F\langle X \rangle$ (where X is an alphabet) we use yet another well ordering. Below we write out this well ordering and the corresponding set of Knuth-Bendix rules. We refer the reader to [39] for basic results on the Knuth-Bendix procedure and use the notation introduced in the paper.

Let $X = \{x_1, \dots, x_n\}$. We introduce a well ordering on the free monoid $M(X \cup X^{-1})$ which induces a well ordering of the free group $F\langle X \rangle$, denote it \prec , as follows.

Definition. Let f and g be two words from $M(X \cup X^{-1})$.

$$f = u_1 x_1^{\epsilon_1} \dots u_k x_1^{\epsilon_k} u_{k+1} \quad g = u'_1 x_1^{\epsilon'_1} \dots u'_{k'} x_1^{\epsilon'_{k'}} u'_{k'+1}$$

where $\epsilon_i, \epsilon'_i \in \mathbb{Z}$, $u_j, u'_j \in M(X \setminus \{x_1\} \cup X^{-1} \setminus \{x_1^{-1}\})$. The word f is less than the word g , write $f \prec g$, if

- $|f| < |g|$;
- $|f| = |g|$, $k < k'$;

- $|f| = |g|$, $k = k'$, $(\epsilon_1, \dots, \epsilon_k)$ precedes $(\epsilon'_1, \dots, \epsilon'_k)$ lexicographically;
- $|f| = |g|$, $k = k'$, $(\epsilon_1, \dots, \epsilon_k) = (\epsilon'_1, \dots, \epsilon'_k)$ and $u_1 \prec u'_1$ or $u_1 = u'_1$ and $u_2 \prec u'_2$ and so on. Roughly saying that is (u_1, \dots, u_{k+1}) precedes (u'_1, \dots, u'_{k+1}) lexicographically. Note that here we assume order \prec to be defined for $M(X \setminus \{x_1\}) \cup X^{-1} \setminus \{x_1^{-1}\}$.

One can prove the following

Proposition. *Relation \prec is a linear ordering of $F\langle X \rangle$. Moreover the order \prec is a well-ordering.*

Let w° be the \prec -minimal representative in the class of all words $[w]$ equal to w in $F\langle X \rangle$.

Define Knuth-Bendix rules on the set $F\langle X \rangle$ by the following recurrent relation:

$$(12) \quad \begin{aligned} KB_n = & KB_{n-1}(x_2, \dots, x_n) \cup \{x_i^\eta w x_1^\varepsilon \rightarrow w x_1^\varepsilon x_i^\eta, i \neq 1; x_1, x_i \notin \alpha(w); \\ & [x_i, \alpha(w x_1)] = 1; \varepsilon, \eta \in \{1, -1\}\} \cup \{x_1^\varepsilon x_1^{-\varepsilon} \rightarrow 1, \varepsilon \in \{1, -1\}\}, \end{aligned}$$

where w is written in the normal form.

Proposition. *The set KB_n is a complete set of rules for the group \mathbb{G}_r .*

Proof. We use the k -completeness criterion from [39]. We shall verify the two clauses of the criterion. We first treat the case when two left sides of the rules overlap.

There are four cases to consider:

- (1) $x_{i_1}^{\varepsilon_1} w_1 x_{i_2}^{\varepsilon_2} w_2 x_{i_3}^{\varepsilon_3}$, where $i_1 > i_2$, $x_{i_1}, x_{i_2} \notin \alpha(w_1)$, $[x_{i_1}, \alpha(w_1 x_{i_2}^{\varepsilon_2})] = 1$, $i_2 > i_3$, $x_{i_2}, x_{i_3} \notin \alpha(w_2)$, $[x_{i_2}, \alpha(w_2 x_{i_3}^{\varepsilon_3})] = 1$, $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{1, -1\}$;
- (2) $x_{i_1}^{\varepsilon_1} w_1 x_{i_2}^{\varepsilon_2} w_2 x_{i_3}^{\varepsilon_3} w_3 x_{i_4}^{\varepsilon_4}$, where $i_1 > i_3$, $x_{i_1}, x_{i_3} \notin \alpha(w_1 x_{i_2}^{\varepsilon_2} w_2)$, $[x_{i_1}, \alpha(w_1 x_{i_2}^{\varepsilon_2} w_2 x_{i_3}^{\varepsilon_3})] = 1$, $i_2 > i_4$, $x_{i_2}, x_{i_4} \notin \alpha(w_2 x_{i_3}^{\varepsilon_3} w_3)$, $[x_{i_2}, \alpha(w_2 x_{i_3}^{\varepsilon_3} w_3 x_{i_4}^{\varepsilon_4})] = 1$, $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in \{1, -1\}$;
- (3) $x_i^{-\varepsilon} w x_j^\eta$, where $j < i$, $x_i, x_j \notin \alpha(w_1)$, $[x_i, w x_j^\eta] = 1$, $\varepsilon, \eta \in \{1, -1\}$;
- (4) $x_i^\varepsilon w x_j^\eta x_j^{-\eta}$, where $j < i$, $x_i, x_j \notin \alpha(w_1)$, $[x_i, w x_j^\eta] = 1$, $\varepsilon, \eta \in \{1, -1\}$.

We use induction on the number of generators to verify case 1.

$$\begin{array}{ccccc}
& & w_1 x_{i_2}^{\varepsilon_2} x_{i_1}^{\varepsilon_1} w_2 x_{i_3}^{\varepsilon_3} & \longrightarrow & w_1 x_{i_1}^{\varepsilon_1} w_2 x_{i_3}^{\varepsilon_3} x_{i_2}^{\varepsilon_2} \\
& \nearrow & & & \downarrow \tau \\
x_{i_1}^{\varepsilon_1} w_1 x_{i_2}^{\varepsilon_2} w_2 x_{i_3}^{\varepsilon_3} & & & & t \\
& \searrow & x_{i_1}^{\varepsilon_1} w_1 w_2 x_{i_3}^{\varepsilon_3} x_{i_2}^{\varepsilon_2} & \xrightarrow{\sigma} & \\
& & & &
\end{array}$$

Transformations τ and σ fall under the inductive assumption, for $x_{i_2} \notin \alpha(x_{i_1}^{\varepsilon_1} w_1)$, and $x_{i_1}^{\varepsilon_1} w_1$ equals $w_1 x_{i_1}^{\varepsilon_1}$ in \mathbb{G}_r .

In case (2) write

$$\begin{array}{ccc}
x_{i_1}^{\varepsilon_1} w_1 x_{i_2}^{\varepsilon_2} w_2 x_{i_3}^{\varepsilon_3} w_3 x_{i_4}^{\varepsilon_4} & \rightarrow & x_{i_1}^{\varepsilon_1} w_1 w_2 x_{i_3}^{\varepsilon_3} w_3 x_{i_4}^{\varepsilon_4} x_{i_2}^{\varepsilon_2} \\
\downarrow & & \downarrow \\
w_1 x_{i_2}^{\varepsilon_2} w_2 x_{i_3}^{\varepsilon_3} x_{i_1}^{\varepsilon_1} w_3 x_{i_4}^{\varepsilon_4} & \rightarrow & w_1 w_2 x_{i_3}^{\varepsilon_3} x_{i_1}^{\varepsilon_1} w_3 x_{i_4}^{\varepsilon_4} x_{i_2}^{\varepsilon_2}
\end{array}$$

Consider the third case

$$\begin{array}{ccc}
x_i^{-\varepsilon} x_i^{\varepsilon} w x_j^{\eta} & \rightarrow & x_i^{-\varepsilon} w x_j^{\eta} x_i^{\varepsilon} \\
\downarrow & & \downarrow \\
w x_j^{\eta} & \leftarrow & w x_j^{\eta} x_i^{-\varepsilon} x_i^{\varepsilon}
\end{array}$$

Similarly in the fourth case we get

$$\begin{array}{ccc}
x_i^{\varepsilon} w x_j^{\eta} x_j^{-\eta} & \rightarrow & w x_j^{\eta} x_i^{\varepsilon} x_j^{-\eta} \\
\downarrow & & \downarrow \\
w x_i^{\varepsilon} & \leftarrow & w x_j^{\eta} x_j^{-\eta} x_i^{\varepsilon}
\end{array}$$

We now verify the second clause of the k -completeness criterion, i.e. the case when one of the left sides of a rule is a subword of another one. There are two cases to consider.

- $x_{i_1}^{\varepsilon_1} w_1 x_{i_2}^{\varepsilon_2} w_2 x_{i_3}^{\varepsilon_3}$, where $i_1 > i_3$, $x_{i_1}, x_{i_3} \notin \alpha(w_1 x_{i_2}^{\varepsilon_2} w_2)$, $[x_{i_1}, \alpha(w_1 x_{i_2}^{\varepsilon_2} w_2 x_{i_3}^{\varepsilon_3})] = 1$, $i_2 > i_3$, $x_{i_2}, x_{i_3} \notin \alpha(w_2)$, $[x_{i_2}, \alpha(w_2 x_{i_3}^{\varepsilon_3})] = 1$, $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{1, -1\}$;
- $x_{i_1}^{\varepsilon_1} w_1 x_{i_2}^{\varepsilon_2} w_2 x_{i_3}^{\varepsilon_3}$, where $i_1 > i_3$, $x_{i_1}, x_{i_3} \notin \alpha(w_1 x_{i_2}^{\varepsilon_2} w_2)$, $[x_{i_1}, \alpha(w_1 x_{i_2}^{\varepsilon_2} w_2 x_{i_3}^{\varepsilon_3})] = 1$, $i_1 > i_2$, $x_{i_1}, x_{i_2} \notin \alpha(w_1)$, $[x_{i_1}, \alpha(w_1 x_{i_2}^{\varepsilon_2})] = 1$, $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{1, -1\}$;

In the first case we have

$$\begin{array}{ccc}
x_{i_1}^{\varepsilon_1} w_1 x_{i_2}^{\varepsilon_2} w_2 x_{i_3}^{\varepsilon_3} & \rightarrow & w_1 x_{i_2}^{\varepsilon_2} w_2 x_{i_3}^{\varepsilon_3} x_{i_1}^{\varepsilon_1} \\
\downarrow & & \downarrow \\
x_{i_1}^{\varepsilon_1} w_1 w_2 x_{i_3}^{\varepsilon_3} x_{i_2}^{\varepsilon_2} & \rightarrow & w_1 w_2 x_{i_3}^{\varepsilon_3} x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2}
\end{array}$$

Consider the second case.

$$\begin{array}{ccc}
x_{i_1}^{\varepsilon_1} w_1 x_{i_2}^{\varepsilon_2} w_2 x_{i_3}^{\varepsilon_3} & \rightarrow & w_1 x_{i_2}^{\varepsilon_2} x_{i_1}^{\varepsilon_1} w_2 x_{i_3}^{\varepsilon_3} \\
\downarrow & & \downarrow \\
w_1 x_{i_2}^{\varepsilon_2} w_2 x_{i_3}^{\varepsilon_3} x_{i_1}^{\varepsilon_1} & = & w_1 x_{i_2}^{\varepsilon_2} w_2 x_{i_3}^{\varepsilon_3} x_{i_1}^{\varepsilon_1}
\end{array}$$

Therefore we checked all the conditions of the k -completeness criterion. \square

Moreover the ordering \prec has the following interesting property.

Proposition. *For any element $w \in \mathbb{G}_r$, the normal form w° and the HNN-normal form $n(w)$ coincide.*

Proof. We use induction on r and show that any word written in the normal form w° is in the normal form $n(w)$. Let

$$n(w) = s_0 x_1^{\alpha_1} s_1 x_1^{\alpha_2} s_2 \dots s_{k-1} x_1^{\alpha_k} v, \quad w^\circ = u_1 x_1^{\varepsilon_1} \dots u_l x_1^{\varepsilon_l} u_{l+1}.$$

If we treat $n(w)$ and w° as elements of an HNN-extension we get $k = l$, $\alpha_i = \varepsilon_i$ and s_i, u_i represent the same coset of A in \mathbb{G}_{r-1} .

Consider the word $n(w)$. Suppose that $n(w) \neq w^\circ$. Then $n(w)$ is not the \prec minimal word in $[w]$. Thus, since (12) is a complete set of rules for the order \prec there exists an interval $n(w)[l, m]$ of $n(w)$ which coincides with a left part of a rule ρ from (12). By the inductive assumption, the rule ρ does not lie in the set $KB_{r-1}(x_2, \dots, x_r)$, more precisely for s_i and v in $n(w)$ holds:

$$s_i^\circ \simeq n(s_i), v^\circ \simeq n(v).$$

Clearly, the rule ρ can not be a cancellation. Consequently, ρ has the form:

$$\begin{aligned} \rho = & (x_i^\eta w x_1^\varepsilon \rightarrow w x_1^\varepsilon x_i^\eta), \text{ where } i \neq 1; \\ & x_1, x_i \notin \alpha(w); [x_i, \alpha(w x_1)] = 1; \varepsilon, \eta \in \{1, -1\}, \end{aligned}$$

i.e. the letter x_i from some representative s can be moved to the rightmost position by the means of commutativity relations of G_r . This derives a contradiction with the condition for a word to be in the HNN-normal form. \square

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E. S. ESYN, I. V. KAZACHKOV, V. N. REMESLENNIKOV: OMSK BRANCH OF MATHEMATICAL INSTITUTE SB RAS, 13 PEVTSOVA STREET, OMSK 644099, RUSSIA